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Stability of Quasilinear Dynamic Systems with after Effect

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The work is devoted to development of mathematical apparatus which facilitates analysis of stability of control systems with aftereffect including analytical methods and numerical algorithms for solution of stability problems by the first non-linear approximation and problems of robust stability for these systems. The article discusses the issues of existence and stability of stationary modes in dynamic systems with aftereffect, as well as influence of external limited impacts on these modes. Criteria of stability have been obtained based on the first non-linear approximation of quasilinear systems with aftereffect and systems with aftereffect with the degree of non-linearity higher than the first degree.

Key words: Functional, Constant, Condition, Inequality, System, solution, integral equation, structural minimization.

Keys issues of up-to-date development of science

One of the key issues of current stage of development of science, engineering and technology includes fundamental studies in the field of simulation, control, qualitative and quantitative analysis of dynamics of complex systems. It is necessary to develop new qualitative and quantitative methods of investigation into the behavior of solution of dynamic systems, composing of program control, searching for conditions of steady, reliable and safe operation of complex dynamic systems with various peculiarities. Currently the creation of new technologies, complex data and engineering systems cannot exist without development of fundamental science in various disciplines.

Analysis of trends of science development, existing scientific publications and subjects of international scientific forums decisively state that the top-priority tasks of humanity in the 21st century are, for instance, as follows:

a) Creation of innovative space technologies and spacecraft systems;
b) Creation of non-conventional power engineering technologies, including gas and oil reprocessing;
c) Creation of global dynamic communication system based on satellite and laser systems;
d) Global solution of transportation problem;
e) Creation of innovative biotechnologies aimed at solution of food problem;
f) Creation of multifunctional flexible automated systems.

Solution of the considered problems cannot be implemented without serious analysis and development of mathematical models and methods of investigation into dynamics of operation of complex systems with consideration for reliability and safety, study of mutual interrelations between fault tolerance and efficiency and so on with various

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peculiarities and conditions.

Brief history of development of mathematical sciences

The first meaningful mathematical results obtained within research into operation of dynamic systems should be, portably, attributed to Isaac Newton. He was the first who clearly formulated and set direct and inverse problems for mathematics: it is required to determine motion on the basis of known initiating forces, and vice versa, it is required to determine forces on the basis of the resultant motion. The latter problem constitutes broadly the problem of searching of control initiating the required motion. The second problem, as known, was successfully solved by Newton on the basis of observations by Tycho Brahe with adjustment for refraction by Johannes Kepler, as well as on the hypotheses of central force by Christiaan Huygens, that is, he discovered the law of universal gravitation. In addition, he succeeded to solve several differential equations describing linear motion of point mass under the action of various forces. On the basis of synthetic geometrical construction Newton successfully described rigid body dynamics in central force field. However, Newton did not provide differential equations and their integrals in analytical form, and his method of successive approximations gave solution of the considered problems in the form of exponential series.

Another significant result, which promoted development of mathematical methods of investigation into differential system, was the development of quadrature method for solution of differential equations by Leibnitz and his successors, Bernoulli brothers. He was the first who proposed the term "differential equations", methods of substitution and integrating factor for solution of certain classes of differential equations.

Significant contribution to further development of the theory of differential equations was made by Leonhard Euler, who gave complete solution of linear non-uniform equation with constant coefficients and who elaborated the integrating factor. In addition, important contribution to the methods of solution of differential equations was made by his contemporaries: Alexis Claude Clairaut, Jean Le Rond D'Alembert, Joseph Louis Lagrange, Brook Taylor.

A crucial point in the investigation into non-linear dynamic systems was the theorem of unique existence, proved by Augustin Louis Cauchy using the Euler method and proposed by him method of successive approximations. In addition, the works by Jacques Charles François Sturm and Joseph Liouville should be mentioned, which originated the investigations into boundary problems.

A significant breakthrough within the study of dynamic systems was development of qualitative theory of differential equations, simultaneously developed by Henri Poincaré (Bellman R. and Cooke K.L., 1963) and A. M. Lyapunov (Barbashin E. A., 1970; Ermolin V.S. and Vlasova T.V., 2014). The methods developed in the scope of this theory made it possible to judge the behavior of differential equations and their peculiarities by the right-hand side of these equations. Significant results in this field were obtained also by George David Birkhoff.

Subsequent development of the research methods of behavior of dynamic systems was carried out in the scope of oscillation theory, deeply developed by such researchers as A. N. Krylov, A. A. Andronov, A. A. Markov, N. P. Erugin, V. I. Zubov, Yu.A. Mitropol'skii' (Afanasiev V. N. et al., 2003.). At the same time the methods of description of dynamic systems were developed as well as mathematical methods of their investigation (Diveev A. et al., 2013; Mikheev S.E., 2008, 2014), mathematical language was being improved both for description and for subsequent study (Bellman R., 1970; Gantmacher F.R. 2000).

Intensive development of engineering in the mid-twentieth century, and especially of automated control systems, created a class of new tasks in the field of the theory dynamic systems: obtaining of solutions of dynamic systems, satisfying various boundary conditions; investigation into stability of solutions of dynamic systems with aftereffect; construction of programmable controls and motions, satisfying boundary and initial conditions; synthesis of these controls; solution of problems of stabilization of program motion in the case of direct and indirect regulation (Barbashin E. A. 1967 ; Tou J., 1964; Barabanov A. T., 1988).

Current trends of development of system analysis of controllable systems dynamics

At present the development of methods of system analysis of dynamics of controllable systems with aftereffect is stipulated both with wide scope of applied problems, among which the major problems consist of control of complex engineering units and technological processes, and with intensive development of computer hardware. New emerging capabilities of computer application, development of hard- and software, data acquisition systems based in microprocessors in the control tasks make it possible to review the existing and to create new analytical qualitative and quantitative research methods of systems with aftereffect of high practical value. These methods facilitate implementation of more accurate forecasting of operation of these systems and determination of limits of dynamic safety of this operation at all stages of life cycle (from design stage to operation stage).

The article propose recursive procedures of determination of two cases of position of equation roots of characteristic polynomial on complex plane and their multiplicity with exact detection of simple stability, that is, existence of periodic oscillations around the equilibrium state in linear system.

In practice the coefficients of linear system can vary as a function of time and space coordinates. In this case two problems appear: searching for analytical criteria of stability of such systems and searching for boundaries of such stability, that is, solution of the problem of robust stability. Such analytical criterion will be established below, and solution of the problem of robust stability will be proposed in the subsequent part. In general case it is demonstrated that the systems with aftereffect on the final time intervals possess the properties of stability on the basis of initial data and within permanently acting perturbations; herewith, the estimations of these influences have been obtained, similar to the estimations for the systems of ordinary differential equations.

Iterative methods of solutions of boundary conditions

Suppose we are given a quasilinear dynamic system with aftereffect

$$\dot{X} = P(t)X + F(t) + \mu S(t, X(t+\theta)) \quad ...(1)$$

and quasilinear boundary conditions:

$$B_1 \le \int_0^T dG(\cdot)X(\cdot) + \mu K(X(\theta)) \le B_1 \quad ...(2)$$

where the components of vector-valued functionals $S(t,\Phi(\theta))$ and $K(\Phi(\theta))$ are real and continuous functionals, determined at $\theta \in [-h,0], \Phi(\theta) \in C[0,T], t \in [0,T]; h > 0$ is the positive constant, μ is the small parameter.

In addition, suppose these functionals satisfy the Lipschitz conditions, so that the inequalities are valid:

$$\| S(t,\Psi_1) - S(t,\Psi_2) \| \le L_1 \| \Psi_1 - \Psi_2 \|_0^{-h}, \| K(\Psi_1) - K(\Psi_2) \| \le L_2 \| \Psi_1 - \Psi_2 \|_0^{-h},$$

where $\Psi_i = (\varphi_{i1}, ..., \varphi_{in})^*$, L_1 and L_2 are the positive constants

$$\| W_1 - W_2 \|_0^{-h} = \max_t \sup_{\theta \in [-h,0]} | w_{1i}(\theta) - w_{2i}(\theta) |$$

.

By analogy, let us introduce supplementary boundary conditions:

$$\int_0^T dG(\cdot)X(\cdot) + \mu K(X(t+\theta)) = \Gamma \quad ...(3)$$

Let us assume that the solution of quasilinear dynamic system (1) $X = X(t)$ satisfies the boundary conditions (2), that is, the system (1) at certain initial function $X(\theta) = \Phi(\theta), \theta \in [-h,0]$ has solution $X = X(t)$, which satisfies supplemental boundary conditions (3) at $\Gamma = \Gamma_0$. Using the Cauchy equation, let us substitute as previously this solution into the boundary conditions (3) at $\Gamma = \Gamma_0$, then we obtain the identity:

$$\int_0^T dG(\cdot)Z(\cdot)X_0 + \int_0^T dG(\tau)Z(\tau)\int_0^{\tau} Z^{-1}(\theta)F(\theta)d\theta +$$
$$+\mu\int_0^T dG(\cdot)Z(\theta)\int_0^T Z^{-1}(\tau)S(\tau, X(\tau))d\tau + \mu K(X(t+\cdot)) = \Gamma_0 \quad ...(4)$$

The identity (4) can be rewritten as follows:

$$A_1 X_0 + H + \mu H_1(X(\cdot)) = \Gamma_0$$

$$X_0 = \Phi(0), \quad H_1 = \int_0^T dG(\theta) Z(\theta) \int_0^\theta Z^{-1}(\tau) S(\tau, X(\tau + \cdot)) d\tau + K(X(\cdot)) \quad ...(5)$$

.

From here it follows that if the set (1), (2) has a solution, then it satisfies the integral equations (5) and the Cauchy equation:

$$X(t) = Z(t)\{X_0 + \int_0^t Z^{-1}(\tau)(F(\tau) + \mu S(\tau, (\tau + \cdot)))d\tau\} \quad ...(6)$$

Theorems of solution existence

Let us consider solution of the problem (1), (3) at various ranks of the matrix A_1. For the case $n \geq m$ the following theorems are valid.

Theorem 1

Let the $(n \times n)$ matrix is invertible. Then for any real vector $\Gamma = \Gamma_0$ there exists $\mu_0 > 0$ such that for any μ, satisfying the condition $|\mu| < \mu_0$, there is a set of solutions of the problem (1), (3) at $\Gamma = \Gamma_0$, whereas some of these solutions can be obtained as the limit of successive approximations.

Proof. Let us explain the main idea of the theorem proof. Presenting solution of the problem (1), (3) $\Gamma = \Gamma_0$ at in the Cauchy form and excluding X_0, using the identity (6) we have that it satisfies the respective set of integral equations:

$$X(t) = X_0(t) + \mu H_2(t, X(t+\theta)) \quad ...(7)$$

where $X_0(t)$ is the solution of the linear problem (1), (2) at $\Gamma = \Gamma_0$. It should be mentioned that $H_2(t, X(t+\theta))$ is the real continuous functional satisfying the Lipschitz conditions (Aizerman M.A. and Gantmacher F.R., 1964) for all its arguments except for the first one:

$$H_2 = -Z(t) A_1^{-1} H_1(X(\cdot)) + Z(t) \int_0^t Z^{-1}(\tau) S(\tau, X(\tau + \cdot)) d\tau .$$

Let us arrange the successive approximations using the equation:

$$X_{k+1}(t) = X_0(t) + \mu H_2(t, X_k(t+\theta)), \quad k = 0,1,2,... \quad (8)$$

$t \in [0,T], X_k(\theta) = X_0, \theta \in [-h,0]$ It should be mentioned that the initial function at such successive approximations can be given by various methods (for instance: $X_k(\theta) = X_0(1 - q\theta)$, where q is a certain constant.

It can be obviously seen that $\| X_0(t) \| < m_1, m_1 > 0 - const$ at $t \in [0,T]$. Let $m_2 > m_1$ is a number. Since the functional H_2 is a continuous function of its arguments, then at any continuous function X(t), such that $\| X(t) \| < m_2, t \in [-h,T]$, we have $\| H_2(t, X(t+\cdot)) \| \leq m_3$, where m_3 is some positive constant.

Let us select a number μ_1, such that at any $|\mu| \leq \mu_1$ the inequality $\| X_0(t) + \mu H_2(t, X(t+\cdot)) \| \leq m_2$ is valid, where $t \in [0,T]$, 0 $X(t)$ is any continuous function, such that $\| X_0(t) \| \leq m_2$ at $t \in [-h,T]$. For this aim it is sufficient to set $\mu_1 = (m_2 - m_1) \backslash m_3$. From here it follows that all approximations (8) are continuous and confined functions at $|\mu| \leq \mu_1, t \in [-h,T]$.

Now let us demonstrate that there exists the number $\mu_0 \leq \mu_1$ such that the series $X_0(t) + (X_1(t) - X_0(t)) + (X_2(t) - X_1(t)) + ...$ uniformly converges in the interval $[-h,T]$ at any μ, satisfying the condition $|\mu| \leq \mu_0$.

Since the functionals S and K satisfy the Lipschitz conditions, then it can be easily demonstrated that there exists such positive value L that

$$\| H_2(t, X(t+\cdot)) - H_2(t, Y(t+\cdot)) \|_0^T \leq L \| X(\cdot) - Y(\cdot) \|_{-h}^T, \quad \mu \leq \mu_1.$$

Let us subtract from the inequality (8) at *k=m* the same inequality at *k=m-1*, then we obtain:

$$\| X_{m+1}(t) - X_m(t) \| \leq \mu L \| X_m(\cdot) - X_{m-1}(\cdot) \|_{-h}^T .$$

From here follows the apparent non-equality:

$$\| X_{m+1}(\cdot) - X_m(\cdot) \|_{-h}^T \leq (\mu L)^m m_2$$

From this estimation it follows that the aforementioned series converges uniformly at $t \in [-h,T]$ and any choice $\mu : |\mu| < \mu_2$

, where $\mu_2 L = 1$. At μ, satisfying the condition $|\mu| \leq \mu_0 = \min(\mu_1, \mu_2)$, construction of successive approximations is possible, each function $X_m(t)$ is continuous and the sequence $\{X_m(t)\}, m = 1,2,...$ converges uniformly in the interval $[-h, T]$.

Denoting the limit of this sequence by $X(t)$, by means of direct differentiating it is possible to discover that this function is a solution of the system (1), and taking into account that it also satisfies the equality (4), then this function is a solution of the problem (1), (3) at $\Gamma = \Gamma_0$. The theorem is proved.

Under the hypothesis of the theorem we can find various solutions of the problem (1), (3) at $\Gamma = \Gamma_0$, using the approach of successive approximations (8) and various ways of setting the initial functions $\Phi(\cdot) = X_m(\cdot), \theta \in [-h, 0]$.

Theorem 2. Let the order of the matrix A_1 is *m*. Then for any real vector $\Gamma = \Gamma_0$ there exists $\mu_0 > 0$ such that for any μ, satisfying the condition $|\mu| \leq \mu_0$, there there is a set of solutions of the problem (1), (3) at $\Gamma = \Gamma_0$, depending on ($n - m$) random constants, whereas some of these solutions can be obtained as the limit of successive approximations.

Proof. Let the problem (1), (3) has a solution at $\Gamma = \Gamma_0$. Then it satisfies the identity (5).

Without restricting the generality let us consider that the first *m* rows of the matrix A_1 are linearly independent and form the invertible matrix A_2, and the remaining (*n-m*) rows form the matrix A_3. Let us decide the equality (5) with regard to the first *m* components of the vector $X_0 : \overline{X}_0 = (x_1, ..., x_m)^*$, considering that the remaining (*n-m*) components are determined by constant vector $C = (c_{n-m+1}, ..., c_n)^*$. Then we obtain the identity:

$$\overline{X}_0 = A_2^{-1}(\Gamma_0 - H - \mu H_1(X(\cdot)) - A_3 C) \quad ...(9)$$

Therefore, we have demonstrated that the solution of the problem (1), (3) satisfies the integral equation (6) at $X_0 = \{\overline{X}_0, C\}$. Let us arrange the successive approximations as previously:

$$X_{k+1}(t) = \overline{X}_0(t) + \mu H_3(t, X_k(t+\theta)), \quad k = 0,1,2,... \quad ...(10)$$

$t \in [0, T], X_k(\theta) = \overline{X}_0(0), \theta \in [-h, 0]$. Here $\overline{X}_0(t)$ is a solution of the linear problem (1), (2) at $\Gamma = \Gamma_0$, in the form of equation (6), where constant vector $\overline{X}_0$ satisfies the equality (7) at μ=0.

It should be noted that the initial function at such successive approximations can be set by various methods (for instance, setting that $X_k(\theta) = \overline{X}_0(0)(1 - q\theta)$, where $\theta \in [-h, 0], q$ is some constant).

Similarly to the case of theorem 1 is can be demonstrated that there exists $\mu_0>0$ such that at any μ satisfying the condition $|\mu| \leq \mu_0$ the series, determined by equations (10), converges uniformly in the interval [-h,T] to the solution of the problem (1), (3) at $\Gamma = \Gamma_0$. The theorem is proved.

Under the hypothesis of the theorems 1, 2 for any vector $\Gamma_0 \in D$ there exists $\mu_0 > 0$ such that for all $\mu : |\mu| \leq \mu_0$ the solution of the problem (1), (3) at $\Gamma=\Gamma_0$ exists and can be obtained by the approach of successive approximations.

On the other hand, it can be readily demonstrated that if the point Γ_0 is an internal point of the parallelepiped D , then the solution of linear problem at $\Gamma = \Gamma_0, X = X_0(t)$ is obtained by another construction method of solutions of quasilinear problem (1), (3) at $\Gamma=\Gamma_0$ and sufficiently small μ. For this aim it is sufficient to construct the solution of equation (1) $X(0) = X_0(0)$ at and random initial function $\Phi(\theta), \theta \in [-h, 0], \Phi(0) = X_0(0)$. The validity of this statements follows from finiteness of this solution at $t \in [-h, T] : \|X(t)\| \leq m_1$, the identity $A_1 X_0 + H = \Gamma_0$, equation (5) and apparent inequality:

$$\|\mu H_1(X(\theta))\|_{-h}^{T} \leq \mu \overline{L} \|X(\theta)\|_{-h}^{T},$$

where $\overline{L}$ is the Lipschitz constant for functional H_1.

It is obvious that the problem of existence and construction of solution of the problem (1), (3) is equivalent to the problem of determination of such initial function for the system (1), so that the motion of the system (1) with this initial function would satisfy the boundary conditions (3).

Theorem 3. If hyperplane W, described by the equation:

$$\Gamma_0 = H + \sum_{j=1}^{k} \alpha_j \Gamma_j \quad ...(11)$$

has a common internal point Γ_0 with parallelepiped D, then there exists $\mu_0 > 0$ such that at all μ, satisfying the condition $|\mu| \le \mu_0$, the problem (1), (3) has the solution.

The proof of this theorem is completely the same as the proof of theorem 2. in other words, if we select the initial function of the linear problem (1), (3) at $\Gamma=\Gamma_0$ as initial function for the system (1), then at sufficiently small we obtain the solution satisfying the boundary conditions (3).

Study of the stability of solutions of non-linear set of ordinary differential equations with aftereffect

Suppose we are given a non-linear set of ordinary differential equations with aftereffect

$$\dot{X} = A(t, X(t+\theta))X + \mu N(t, X(t+\theta)) \quad (1)$$

where the elements $a_{ij}(t,\Phi(\theta))\,(i,j=1,...,n)$ of the matrix $A(t,\Phi(\theta))$ and the components of the vector are real and continuous functionals, determined and confined at $t \ge 0, \Phi(\theta) \in C[-h,0], \|\Phi(\theta)\|^h < H$ and satisfying the Lipschitz conditions in this region with regard to all its components except for the first one:

$$|a_{ij}(t,\Phi(\theta)) - a_{ij}(t,\Psi(\theta))| < l_{ij} \|\Phi(\theta) - \Psi(\theta)\|^h \;(i,j=1,\dots,n),$$

$$\|A\| = \max_{i,j=1,\dots,n} |a_{ij}|, \|X\| = \sqrt{\sum_{i=1}^{n} x_i^2}, \|\Phi(\theta)\|^h = \max_{\theta \in [-h,0]} \|\Phi(\theta)\|,$$

$l_i, l_{ij}, h, H > 0 - const, \mu -$ is the small parameter.

Since the right-hand part of the system (1) satisfies the local Lipschitz conditions, then for any initial function $\Phi(\theta) \in C[-h,0], \|\Phi(\theta)\|^h \le H_0 < H$ all hypotheses of the theorem of unique existence are satisfied and the solutions of the system (1) are continuable at $t \ge 0$ up till $\|X(t)\| < H$.

For subsequent study we need the following Lemma (Barbashin E. A. and Krasovskii N. N., 1952):

Lemma. Let u(t) is the continuous function satisfying the inequality at $t \ge t_0$ as follows:

$$0 < u(t) < \delta + \int_{t_0}^{t} (\eta + Lu(\tau))\,d\tau \quad ...(2)$$

where δ,η,L are constants, which satisfy the relations: $\delta \ge 0, \eta \ge 0, L > 0$. Then the $\forall t \ge t_0$ following inequality is valid

$$u(t) < \frac{\eta}{L}(\exp(L(t-t_0)) - 1) + \delta \exp(L(t-t_0)) \quad ...(3)$$

Proof. It is obvious that at $t \ge t_0$ the inequality (3) is valid. By virtue of continuity of the function *u(t)* the inequality (3) will be valid also at $t > t_0$ for all $t-t_0 \ge t_0$, where ε<0 is some positive number. Let us assume that the inequality (3) will be violated for the first time in the point t=ξwithin increase in *t* from t_0 to ξ, that is, this inequality transforms into equality in the point t=ξ

$$u(\xi) = \frac{\eta}{L}(\exp(L(\xi-t_0)) - 1) + \delta \exp(L(\xi-t_0)) \quad ...(4)$$

If we replace the function *u(t)* in the inequality (2) at t=ξ with the right-hand part of the inequality (3), which exists in the semi-interval $t_0 \le t < \xi$, then we obtain the following inequality:

$$u(t) < \delta + \int_{t_0}^{\xi} (\eta + L[\frac{\eta}{L}(\exp(L(\tau-t_0)) - 1) + \delta \exp(L(\tau-t_0))])\,d\tau \quad ....(5)$$

After calculation of the integral in the right-hand part of this inequality we obtain the following inequality:

$$u(\xi) < \frac{\eta}{L}(\exp(L(\xi-t_0)) - 1) + \delta \exp(L(\xi-t_0)) \quad ...(6)$$

The inequality (6) is strict and contradicts with the equality (4). from here it follows that the inequality (3) is valid . The lemma is proved $\forall t \ge t_0$.

The following theorem of asymptotic stability is valid (Razumikhin B. S., 1956.).

Theorem 4. Suppose we are given that for the equation there exists a continuous positively defined function $V(t,X), V: R_1 \times R_n \to R_n$, the total derivative of which with time calculate by virtue of this equation is non-positive along any solution X(t), satisfying the condition:

$V(\sigma, X(\sigma)) \le V(t, X(t)), \quad \sigma < t, \quad t \ge t_0$ ...(7)

Then the trivial solution of the considered equation is stable according to Lyapunov. If in addition the function *V(t,X)* admits infinitesimally small upper limit, and its derivative is a function positively defined along any solution satisfying the condition (7), then the trivial solution of the considered equation is asymptotically stable (Afanisev V. N. et al., 2003).

Let $\lambda_i(t)$ are the eigenvalues of the Hermitean-conjugated matrix $B_0(t) = (A(t,0) + A^*(t,0))/2$, where * is the conjugation sign. On the basis of the result by B. S. Razumikhin let us prove the following theorem.

Theorem 5. If the following conditions: 1. $\lambda_i(t) < \lambda_0 < 0$; 2. $f_i(t,0) \equiv 0\ (i = 1,...,n)$ are valid at $t \ge 0$, then there exists a positive number $\mu_0 > 0$ such that for all the $\mu : |\mu| \le \mu_0$ trivial solution of the system (1) $X \equiv 0$ is asymptotically stable.

Proof. Let us consider random continuous initial function $\Phi(\theta) \in C[t_0 - h, t_0], \| \Phi(t+\theta) \|^h \le H_0 < H, t_0 \ge 0$.

Taking into account that the solutions of the system (1) originating in this region are continuable at $t > t_0$, then for any μ these exists some value $\Delta > 0$ such that in the interval $[t_0, t_0 + \Delta]$ there exists the single solution of the system (1) $X = X(t)$ satisfying this initial function and belonging to the definition region $\| X(t) \| < H$. Moreover, by virtue of stability of solutions of the system (1) in the final time interval the value H_0 can be selected so that the inequality Δ>h is valid. Let us demonstrate that at sufficiently small μ and H_0 this solution tends to zero, that is, the trivial solution of the system (1) is asymptotically stable.

Let us introduce a continuous $(n \times n)$ matrix for consideration:

$$B(t) = (A(t, X(t+\theta)) + A^*(t, X(t+\theta)))/2$$

set in this solution at $t \in [t_0, t_0 + \Delta]$. Since the elements $a_{ij}(t, \Phi(\theta))\ (i,j = 1,...,n)$ of the matrix $A(t, \Phi(\theta))$ are real and continuous functional, determined at $t \ge 0, \Phi(\theta) \in C[-h, 0], \| \Phi(\theta) \|^h \le H$ and satisfying the Lipschitz conditions in this region with regard to all its components except for the first one, then by virtue of continuous dependence of eigenvalues of the matrix *B(t)* on coefficients of its characteristic polynomial, and, hence, on its elements, there exists a value $H_1 \le H$, such that at $t \ge 0, \| \Phi(\theta) \|^h \le H_1$ the inequalities $\lambda_i(t, \Phi) \le \lambda_1 < \lambda_0 < 0 \quad (i = 1,...,n)$ are valid. Let us take a random initial function $\Phi(t) \in C[t_0 - h, t_0], \| \Phi(\theta) \|^h \le \min(H_0, H_1) t_0 \ge 0$.

By virtue that the matrix *B(t)* is Hermitean-conjugated matrix, that is, $B(t) = B^*(t)$, there exists an invertible continuous orthogonal matrix *S(t)* such that the matrix can be presented in the form of $B(t) = S(t)\Lambda(t)S^{-1}(t)$, where $\Lambda(t)$ is the diagonal matrix, the eigenvalues of which $\lambda_i(t, X)$ satisfy the inequality $\lambda_i(t, X) \le \lambda_1\ (i = 1,...,n)$, and $S^*(t) = S^{-1}(t)$. it should be mentioned that the matrix $S(t)$ is confined at $t \ge 0$, since the matrix $B(t)$ is confined at $t \ge 0$.

Let us consider the function $Y(t) = S^*(t)X(t)$. It is obvious that the following equality is valid: $z(t) = \| Y(t) \|^2 = Y^*(t)Y(t) = X^*(t)S(t)S^*(t)X(t) = \| X(t) \|^2$.

Let us take the scalar function *z(t)* as the Lyapunov function, that is $V(t,X) = X^*X$, and estimate its total derivative by virtue of the equation (1) along any solution of this equation satisfying the condition (2): $X^2(t_1) \le X^2(t_2), t_1 < t_2$. It is obvious that these solutions satisfy also the conditions $\| X(t_1) \| \le \| X(t_2) \|, t_1 < t_2$ and, hence $\| X(t) \| = \sup_{\theta \in [-h,0]} \| X(t) \| = \| X(t) \|^h$. By differentiating the function $V(t,X) = X^2$ by virtue of the system (1) in the interval $[t_0, t_0 + \Delta]$ we obtain:

$$\frac{dV}{dt} = \dot{X}^* X + X^* \dot{X} = 2X^* BX + 2\mu F^*(t, X(t+\theta))X) \quad ...(8)$$

Let us introduce the designations: $L = \max_{i=1,..,n} l_i, \quad M = \max_{t \in [t_0, t_0+\Delta]} \| S(t) \|$.

Using these designations, the Lipschitz conditions and the relation $\| X(t) \| = \| X(t) \|^h$ for the right-hand part of the equation (8) it is possible to write the following inequality:

$$2X^*BX + \mu F^*(t, X(t+\theta))X) \leq 2Y^*\Lambda Y + 2\mu L \| X(t) \|^h \| X(t) \| \leq \quad ...(9)$$

$$\leq 2(\lambda_1 + \mu LM^2) \|Y\|^2$$

If we select $\mu_0 > 0$ so that $\mu_0 = \frac{-\lambda}{LM^2}$, then it follows from the latter inequality and the identity (8) that at all $\mu : |\mu| \leq \mu_0$ the inequality will be valid:

$$\frac{dV}{dt} \leq 2\alpha V(t), \quad \alpha = \lambda_1 + \mu LM^2 < 0$$

Therefore, for the function $V(t, X) = X^* X$ all hypotheses of the B. S. Razumikhin are satisfied, it means that the trivial solution of the system (1) is asymptotically stable.

Moreover, any solution of the system (1) at $|\mu| \leq \mu_0$ with initial function belonging to some vicinity of the origin of coordinates $\| \Phi(\theta) \|^h \leq \min(H_0, H_1)$ tends to stable position of this system $X \equiv 0$. It means that this vicinity belongs to the region of asymptotical stability of the system (1). The theorem is proved.

If the matrix , $A(t, \Phi(\theta)) = A(t)$ that is, it is independent on the phase coordinates and the conditions $\lambda_i(t) \leq \lambda_0 < 0 \ (i = 1, ..., n)$ are valid for the system (1) at $t \geq 0$ N $\mu = 0$, then the set of initial functions $\Phi(\theta) \in C[-h, 0], \| \Phi(\theta) \|^h < H$ belongs to the region of asymptotical stability.

The problem of development of accurate solutions of equations describing operation of dynamic systems is determined, on the one hand, by insufficient accuracy of both the equations and their parameters. The investigation into such fundamental aspects, determining operation of dynamic system, as stability, controllability and observability are considered at certain approximations to the required ones. Thus, instead of accurate solutions of dynamic systems, it is possible to consider their certain approximation, and development of control systems based on stability by Lyapunov (Ermolin V.S., 1995; Ermolin V.S., 1996; Ermolin V.S. et al., 2002) can be replaced by solution of the problem of stabilization of program motion or kinematic path (Azbelev N. V. et al., 2002). Control of engineering subjects with consideration for structural flexibility can be interpreted by means of solution of the control problem of finite-dimensional mechanic systems (Andreev Yu. N., 1976; Andrievskii B. R. and Fradkov A. L., 1999). Within the problem solution of searching for permissible controls (optimal in terms of certain criteria) there appears the problem of structural minimization of both the control system of dynamic system, in particular, and finally of the controllable system itself. The initial works on this topic were performed by (Kalman R.E. et al., 1969), though the term "structural minimization" appeared slightly later, and in this case of practical implementation of the problem solution (1)–(3) it can be applied as a kind of the most common approaches to the control theory of dynamic systems. Further improvement of the methods of structural minimization of dynamic systems, published in (Zubov A.V. et al., 2010; Dikusar V.V. et al., 2010), demonstrates that the substrate approach as one of the most common approaches to the control theory of dynamic systems in general, as well as structural minimization as a part of structural optimization (Vapnik V.N., 1982) in particular, are being developed. Application of similar mathematical simulation methods of dynamic systems can be sufficiently versatile and is implemented in various fields (Grigorieva X. and Malafeev O., 2014; Alferov G.V. et al., 2014).

CONCLUSIONS

The results of this study can be applied in the problems of system analysis of controllable systems described by differential equations, including those with aftereffect; synthesis of control laws and development of efficient numerical algorithms of control theory. Efficient algorithms of stability research of systems of the first approximations can be based on these results, as well as engineering criteria of investigation into the boundaries of this stability in the space of parameters of the considered systems. The obtained applied methods of system analysis make it possible to solve the problem of dynamic safety of a considered subject, that is, to determine permissible variation boundaries of the system parameters with retaining of the stability of the considered system. Herewith, it becomes possible to develop more efficient control systems, since the developed methods facilitate consideration of entire family of mathematical models of operation dynamics of controllable systems, determined by a set of their permissible parameters. This permits significant reduction of material resources, monetary assets and time for adjustment of newly created systems.

On the basis of classical results and methods of qualitative theory of differential equations, theory of stability, functional analysis, mathematical programming, linear and higher algebra this work develops in details a method of system analysis of investigation into operation dynamics of controllable and non-controllable quasilinear systems with aftereffect based on the studies of these systems by the first linear and non-linear approximation. Therefore, the essence of this method is the analysis of influence of dynamic properties of a system of the first approximation on qualitative character of the system behavior in whole.

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